Math 10A
Practice Final; Monday, 8/6/2018
Time: 2:10 PM **Name:** ____________________
Instructor: Roy Zhao **Student ID:** ____________________

- **DO NOT OPEN THE PRACTICE FINAL UNTIL TOLD TO DO SO!**
- Do all problems as best as you can. The exam is 60 minutes long. You may not leave during the last 30 minutes of the exam.
- Use the provided sheets to write your solutions. You may use the back of each page for the remainder of your solutions; in such a case, put an arrow at the bottom of the page and indicate that the solution continues on the back page. **No extra sheets of paper can be submitted with this exam!**
- The exam is closed notes and book, which means: **no class notes, no review notes, no textbooks, and not other materials can be used during the exam.** You can only use your cheat sheet. The cheat sheet is one side of one regular 8×11 sheet, handwritten.
- **NO CALCULATORS ARE ALLOWED DURING THE EXAM!**
- Justify all your answers, include all intermediate steps and calculations, and box your answers.

1. (20 points) Find the following derivatives and integrals.

(a) (5 points) $\frac{d}{dx}(\cos(\cos(x)))$.

Solution: Use the chain rule to get $-\sin(\cos(x))\cdot(-\sin(x)) = \sin(\cos(x))\sin(x)$.

(b) (5 points) $\frac{d}{dx}\int_{\sqrt{x}}^{x^2} e^{x^2}dx$.

Solution: Use FTC to get $e^{(x^2)^2}\cdot 2x - e^{(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}} = 2xe^{x^4} - \frac{e^x}{2\sqrt{x}}$.

(c) (5 points) $\int x\cos(x)dx$

Solution: We use integration by parts since it is a product of functions. Let $u = x$ and $dv = \cos(x)dx$ so $du = dx$ and $v = \sin(x)$. This is

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) - (-\cos(x)) + C = x\sin(x) + \cos(x) + C.$$

(d) (5 points) $\int_{-1}^{0} 2x\sqrt{x+1}dx$.

Solution: First we u sub by $u = x+1$ so $du = dx$. Then $2x = 2(u-1)$ so this integral is

$$\int_0^1 2(u-1)\sqrt{u}du = \int_0^1 2u\sqrt{u} - 2\sqrt{u}du = \int_0^1 2u^{3/2} - 2u^{1/2}du$$

$$= \frac{4u^{5/2}}{5} - \frac{4u^{3/2}}{3}\bigg|_0^1 = \frac{4}{5} - \frac{4}{3} = \frac{-8}{15}.$$

2. (20 points) Sand is being dumped in a conical pile whose width and height always remain the same. If sand is being dumped in at a rate of $\pi cm^3/s$, how fast is the height of the sand changing when the pile is 4 cm tall? (Hint: The volume of a cone with radius r and height h is $V = \frac{\pi r^2 h}{3}$).

Solution: Since the width and height always remain the same, we know that $2r = h$ or $r = h/2$. Therefore, the volume of the cone is given by $V = \frac{\pi (h/2)^2 h}{3} = \frac{\pi h^3}{12}$. Taking the derivative, we get that $V' = \frac{\pi h^2}{4} h'$. So

$$h' = \frac{4V}{\pi h^2} = \frac{4\pi}{\pi (4)^2} = \frac{1}{4} \frac{cm}{s}.$$

3. (20 points) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 0 & 3 & 1 \end{pmatrix}$.

(a) (5 points) What is the determinant of A?

Solution: The determinant is $1\cdot7\cdot1+2\cdot1\cdot0+0\cdot3\cdot3-0\cdot7\cdot0-1\cdot1\cdot3-2\cdot3\cdot1 = 7+0+0-0-3-6=-2$.

(b) (15 points) Use Gaussian elimination to find the solution to $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 3 & 7 & 1 & 2 \\ 0 & 3 & 1 & 1 \end{array}\right) \xrightarrow{II-3I} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 1 & 1 \end{array}\right) \xrightarrow{I-2II,III-3II} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 4 \end{array}\right)$$

$$\xrightarrow{III/-2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{I+2III,II-III} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right).$$

So, the solution is $\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$.

4. (20 points) Find the general solution of the following differential equations.

(a) (10 points) $y'' - 2y' + 5y = 0$.

Solution: The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$ which has roots $1 \pm 2i$. Therefore, the general solution is $y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$.

(b) (10 points) $y' + 3\frac{y}{x} - \frac{e^x}{x^3} = 0$.

Solution: We want to use integrating factors. Moving only the y's on one side, the equation becomes $y' + 3\frac{y}{x} = \frac{e^x}{x^3}$. The integrating factor is $I(x) = e^{\int 3/x dx} = e^{3\ln x} = x^3$. Multiplying this new equation by the integrating factor gives $x^3 y' + 3x^2 y = e^x$. Finally, integrating gives

$$(x^3 y) = \int e^x = e^x + C.$$

So the general solution is $y = \frac{e^x + C}{x^3}$.

5. (20 points) (a) (15 points) Find the line of best fit through the set of points $\{(1,1),(2,0),(3,5)\}$.

Solution: Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$. Then $A^T A = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}$ and $(A^T A)^{-1} A^T \vec{y} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$. So the line of best fit is $y = 2x - 2$.

(b) (5 points) Calculate the least square error of using your line of best fit from above.

Solution: The expected y values were in order $2(1)-2 = 0, 2(2)-2 = 2, 2(3)-2 = 4$. So the least square error is $(\text{expected}-\text{actual})^2 = (0-1)^2 + (2-0)^2 + (4-5)^2 = 1+4+1 = 6$.